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# Faces of the Gomory Polyhedron for Cyclic Groups*

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A two parameter method is given for generating faces of Gomory's master polyhedron for cyclic groups. The paper shows how to specify coefficients of the linear inequality defining a face directly without requiring a special group minimization algorithm. Included are results which specify a unique representation of a cyclic group in terms of restricted multiples of an appropriately selected pair of elements.

A computer study indicates that the two parameter method is very efficient, requiring roughly the same amount of computation as Gomory's "$\theta$-method," but capable of generating a substantially larger number of faces.

## 1. INTRODUCTION

We show how to generate faces of Gomory's integer polyhedron [7, 8] to provide cuts for the integer program

$$\text{Min}\{c'z \mid z \in S'\} \qquad S' = \{z \mid A'z = b',\, z \geqslant 0 \text{ and } z \in J^m\} \tag{1}$$

where $A'$, $b'$ and $c'$ are integer matrices and $J^m$ is the set of integer $m$ vectors.

We assume $A'$ initially has the form $(A^0I)$. By a permutation of columns of $A'$ and components of $z$ and $c'$ we write $A' = (NB)$, $z = \binom{x}{y}$, $c' = (c_x, c_y)$ where $B$ is a basis for $A'$. The typical process of solving (1) as a linear program (disregarding the restriction $z \in J^m$) leads to identifying such a basis $B$ and obtaining the equivalent representation

$$\text{Min}\{cx \mid x, y \in S\}$$
$$S = \{x, y \mid Ax + Iy = b,\, x \geqslant 0,\, y \geqslant 0,\, x \in J^n,\, y \in J^{m-n}\} \tag{2}$$

where $A = B^{-1}N$, $b = B^{-1}b'$, $c = c_x - c_yA$ (the constant $-c_yb$ of the objective function in (2) is ignored).

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We suppose the representation (2) is primal and dual feasible (i.e., $b \geqslant 0$, $c \geqslant 0$) and hence optimal in the linear programming sense.

Cutting approaches to integer programming customarily specify an additional set of constraining relations, or *cuts*, summarized by the matrix inequality $Qz \geqslant d$, where $\{z \mid Qz \geqslant d\} \supset S'$ (i.e., the lattice points of $S'$ are feasible for $Qz \leqslant d$). In addition, the cuts are desired to be restrictive enough that the optimal integer solution to (1) can be obtained by replacing $z \in J^m$ with $Qz \geqslant d$ in $S'$ and solving the resulting linear program.

Of the various inequalities that may be represented by $Qz \geqslant d$, the most restrictive are the $m$-dimensional faces of the convex hull of $S'$. Unfortunately, the task of specifying such an ideal set of cuts appears to be extremely difficult. It is easier, seemingly, to try to solve (1) directly (e.g., by an iterative algorithm that generates a succession of weaker cuts).

A related, but less difficult goal, can be pursued via the representation (2). Suppose $y \geqslant 0$ is no longer required. Then (2) becomes

$$\operatorname{Min}\{cx \mid Ax + Iy = b,\ x \geqslant 0,\ x \in J^n,\ y \in J^{m-n}\}.$$

This corresponds to replacing $S$ in (2) with

$$S^0 = \left\{x \mid \sum_{j=1}^{n} \alpha_j x_j = \alpha_0\,,\ x \geqslant 0,\ x \in J^n\right\}$$

where $\alpha_0$ and $\alpha_j$, $j = 1,..., n$, are elements of a finite additive group and the integer vector $y$ is now given from the definitional equation $y = b - Ax$ (Gomory [5, 6]).

It is tempting to examine the less constrained problem

$$\operatorname{Min}\{cx \mid x \in S^0\} \tag{3}$$

in hopes that solving it may give a key to solving (2) (equivalently, (1)). Indeed, Gomory [6] has specified a dynamic programming recursion for solving (3) and developed properties that link optimal solutions of (3) to those of (2) under certain conditions. Subsequently, other methods for solving (3) have been proposed by White [10], Shapiro [9], Glover [2], and Hu [11], and further properties of optimal solutions given in [2].

However, following Gomory [7], the use of (3) that we consider in this paper is to generate faces of the convex hull of $S^0$ to obtain cuts for (2). While this is a more modest undertaking than attempting to generate faces for $S'$, it nevertheless provides cuts that are considerably stronger than many of those previously specified [4, 5, 1].

Two approaches to this objective have been proposed by Ralph Gomory [7, 8]. The first involves determining basic feasible solutions to a special linear program. Unfortunately, the nature of this linear program makes the generation of faces by this method an exceptionally onerous task. The second approach can be applied to cyclic groups under the assumption that $n$ is equal to the group order (or one less). However, only $n/2$ faces are generated, which generally constitutes a small portion of the total number of faces.

In this paper we give an efficient procedure for generating large numbers of faces for cyclic groups. A sequel to this paper [3] extends its results and gives other methods for generating faces.

## 2. The Group Equation and Cut Face

Let $G = \{g_0, g_1, g_2, ..., g_h\}$ denote an additive group of order $h + 1$, with $g_0 = 0$ (the group identity element). As shown in Gomory [8], the faces for the system

$$\sum_{j=1}^{n} \alpha_j x_j = \alpha_0$$
$$x_j \geqslant 0 \quad \text{and integer}, \qquad j = 1, ..., n \tag{4}$$

where $\alpha_j \in G$, $j = 0, 1, ..., n$ are "contained in" the faces for the system[1]

$$\sum_{j=1}^{h} g_j w_j = \alpha_0$$
$$w_j \geqslant 0 \quad \text{and integer}, \qquad j = 1, ..., h, \tag{5}$$

in a special sense described to follow.

For convenience, suppose $\alpha_j = g_j$ for $j = 1, ..., n$, and let

$$\sum_{j=1}^{h} a_j w_j \geqslant a_0 \tag{6}$$

denote any linear inequality (cut) implied by (5) (where the $a_j$ are scalar constants). Then the inequality (6) is defined to be a face for (5) if there are $h$ linearly independent solutions to (5) that satisfy (6) with equality. Similarly, the inequality

$$\sum_{j=1}^{n} a_j x_j \geqslant a_0 \tag{7}$$

[1] We disregard the trivial generalization that includes $g_0$ in (5) and permits $\alpha_0 = g_0$.

is defined to be a face for (4) if it is implied by (4) and there are $n$ linearly independent solutions to (4) that satisfy (7) with equality. Gomory's result that connects the faces of (4) to those of (5) says that if the $a_j$ coefficients are allowed to vary so that (6) ranges over all faces of (5), then (7) will range over all faces of (4) (plus some cuts that are not faces). Thus Gomory has suggested the indirect strategy of generating faces for (5) as a means of obtaining faces for (4). We pursue this strategy in the following sections.[2]

## 3. The Face Generating Procedure

We suppose $G$ is a cyclic group with $g_u$ one of its generators and $g_v$ any of its other elements ($\neq g_0$). We generate faces by "matching" certain multiples of $g_u$ with certain multiples of $g_v$.

The basic idea is to select a multiple $p$ of $g_u$ and a multiple $q$ of $g_v$ such that $pg_u$ and $qg_v$ are the same group element $g_t$. Then values are assigned to $a_v$ and $a_u$ so that $a_t = pa_u = qa_v$ (just as $g_t = pg_u = qg_v$). Each remaining $a_j$ is then assigned the smallest possible value that can be expressed in the form $a_j = \theta a_u + \delta a_v$, where $\theta$ and $\delta$ are nonnegative integers such that $g_j = \theta g_u + \delta g_v$. We will shortly give rules that provide such values for all $a_j$ (including $a_u$ and $a_v$), without requiring the use of an algorithm for solving the group minimization problem to determine $\theta$ and $\delta$.

### 3.1. *Determining $p$ and $q$*

We first give our attention to the problem of identifying appropriate values of $p$ and $q$ on which the coefficients $a_j$ may be based. Not all $p$ and $q$ such that $pg_u = qg_v$ are permissible. In fact, as $p$ assumes successively larger values, $q$ must assume successively smaller values, according to the following prescription.

First Face. Let $q = 1$ and $p$ the least positive integer such that $pg_u = qg_v$ $(=g_v)$.

Subsequent Faces.

(1) Let $p' = p$ for $p$ given in obtaining the first face.

(2) Let $q$ be one more than its previous value.

(3) If $qg_v = 0$ or $g_u$ [3] there are no more faces to be generated and the procedure stops. Otherwise,

[2] A way to generate faces *directly* for (4) is developed in [3].

[3] One can generate a face for $pg_v = g_u$ if $g_v$ is also a generator, but this face is homomorphically equivalent to the first face generated.

(4) identify the least integer $p$ such that $pg_u = qg_v$. If $p \geqslant p'$ (the value of $p$ for the previous face generated), return to (2). If $p < p'$, determine the coefficients $a_j$ to generate a face (as indicated in the rules of Section 3.2 to follow), set $p' = p$ and return to (2).

*Remark* 1.[4] The rules for determining $p$ and $q$ imply that the only integer solutions to:

$$g_u x_u = g_v x_v,$$
$$0 \leqslant x_u \leqslant p, \qquad 0 \leqslant x_v \leqslant q,$$

are:

$$x_u = p, \quad x_v = q, \qquad \text{and} \qquad x_u = 0, \quad x_v = 0.$$

DEFINITION. Let $p^* = \mathrm{Min}\{k : k \geqslant 1 \text{ and } kg_u = k_0 g_v \text{ for some } k_0 \text{ satisfying } 0 \leqslant k_0 < q\}$. Also, for $p^*$ as defined, let $q^*$ denote the unique integer satisfying

$$0 \leqslant q^* < q \qquad \text{and} \qquad p^* = q^* g_v.$$

*Remark* 2. Remark 1 holds with $p$ and $q$ replaced by $p^*$ and $q^*$. Also $p^* > p \geqslant 0$ and $q > q^* \geqslant 0$.

DEFINITION. Let $p_0 = p^* - p$ and $q_0 = q - q^*$.

*Remark* 3. $p_0 g_u + q_0 g_v = 0$ and $p_0, q_0 \geqslant 1$.

Other observations may also be made concerning the relationships between $p$ and $q$. First, however, we shall specify the rules for determining the coefficients $a_j$ to generate a face.

### 3.2. *Determining the Coefficients $a_j$ (given $p$ and $q$)*

1. Let $a_j = pk$ for $a_j$ corresponding to the group element $g_j = kg_v$, $k = 1, \ldots, q$.

2. Let $a_j = qk$ for $a_j$ corresponding to the group element $g_j = kg_u$, $k = 1, \ldots, p$.

3. If $a_r$ is determined for $g_r = kg_u$, but $a_s$ is *not* determined for $g_s = (k + 1) g_u$, let $a_s = a_r + q$.

4. Let $a_0 = \mathrm{Max}\{a_j\}$ and identify $\alpha_0$ as the corresponding group element. (Specifically, $\alpha_0 = (p - 1) g_u - g_v$).

### 3.3. *An Example*

Before attempting to justify the rules for determining $p$ and $q$ and the coefficients $a_j$, we first illustrate these rules with a numerical example.

[4] We prove only those remarks that do not follow immediately from the definitions or preceding development.

Suppose $G$ is the group of integers modulo 10, and $g_j = j$, $j = 0, 1, \ldots, 9$. We shall determine faces for

$$w_1 + 2w_2 + 3w_3 + \cdots + 9w_9 \equiv \alpha_0 \pmod{10}$$

by selecting the generator $g_u = g_3 = 3$ and the group element $g_v = g_2 = 2$.

Group elements corresponding to the multiples of $g_u = g_3$ and $g_v = g_2$ are given in Table 1.

TABLE 1

| Multiple $k$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $g_j = kg_u$ : | 0 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 |
| $g_j = kg_v$ : | 0 | 2 | 4 | 6 | 8 | 0 | 2 | 4 | 6 | 8 |

The rules of Section 3.1 applied to the table yield values for $(p, q)$ of (4, 1) and (3, 2). Thereupon, the rules of Section 3.2 for generating the $a_j$ yield the following faces.

For $(p, q) = (4, 1)$:

$$7w_1 + 4w_2 + w_3 + 8w_4 + 5w_5 + 2w_6 + 9w_7 + 6w_8 + 3w_9 \geqslant 9 \quad \text{with} \quad \alpha_0 = g_7 = 7.$$

For $(p, q) = (3, 2)$:

$$11w_1 + 2w_2 + 3w_3 + \cdots + 9w_9 \geqslant 11 \qquad \text{with} \quad \alpha_0 = g_1 = 1.$$

Note that the first face can also become a face for $\alpha_0 = g_1$ (for comparison with the second face) by the homomorphism $g_j \to 3g_j$ (since $g_1 = 3g_7$), yielding

$$9w_1 + 8w_2 + 7w_3 + 6w_4 + 5w_5 + 4w_6 + 3w_7 + 2w_8 + w_9 \geqslant 9 \quad \text{with} \quad \alpha_0 = g_1 .$$

To justify the rules for determining both $(p, q)$ and the coefficients $a_j$, we introduce the following results.

## 4. Justification of the Face Generating Procedure

Definition. Let

$$S_j = \{x_u, x_v : g_j = g_u x_v + g_v x_v \text{ and } x_u, x_v \geqslant 0 \text{ and integer.}\}$$

Let $T = \{x_u, x_v : p^* > x_u \geqslant 0, q > x_v \geqslant 0$ and either $p_0 > x_u$ or $q_0 > x_v\}$

$$\delta_j = \min\{qx_u + px_v : x_u, x_v \in S_j\}$$

Our goal is to show that $\delta_j$ is precisely the value assigned to $a_j$ by the rules of Section 3. Moreover, we will show that $\delta_j$ is unchanged by additionally requiring $x_u, x_v \in T$, and in fact, that every element of $G$ is uniquely represented as a linear combination of $g_u$ and $g_v$ by requiring $x_u, x_v \in T$. Using these results we will prove that the coefficients $a_j = \delta_j$ determine a face for (5) with $\alpha_0$ appropriately specified.

Lemma 1. *The value of $\delta_j$ is unchanged by requiring $x_u, x_v \in S_j \cap T$; hence, the entire group $G$ may be generated from multiples of $g_u$ and $g_v$ restricted to $T$, i.e., $G = \{g_j : g_j = g_u x_u + g_v x_v$ for $x_u, x_v \in T\}$.*

*Proof.* We remark first that $\delta_j$ is meaningfully defined. Suppose $\delta_j = q\bar{x}_u + p\bar{x}_v$ where $\bar{x}_u, \bar{x}_v \in S_j$ and $\bar{x}_v \geqslant q$. Let

$$x_v' = \bar{x}_v - q \qquad \text{and} \qquad x_u' = \bar{x}_u - p$$

Then,

$$\delta_j = qx_u' + px_v' \qquad \text{and} \qquad x_u', x_v' \in S_j \tag{*}$$

Moreover, $x_v' < \bar{x}_v$. If now $x_v' \geqslant q$, we may repeat the foregoing process (with $x_u'$ and $x_v'$ in role of $\bar{x}_u$ and $\bar{x}_v$) until eventually $x_u'$ and $x_v'$ are found satisfying (*) and $x_v' < q$. We assert that $x_u'$ and $x_v'$ are then also in $T$. For if $x_u' \geqslant p^*$, one can define $x_u'' = x_u' - p^*$ and $x_v'' = x_v' + q^*$, yielding $x_u'', x_v'' \in S_j$ and $qx_u'' + px_v'' < \delta_j$ (Remark 2), contrary to the definition of $\delta_j$. Similarly, if $x_v' \geqslant q_0$ and $x_u' \geqslant p_0$, then letting $x_v'' = x_v' = x_v' - q_0$ and $x_u'' = x_u' - p_0$ leads to the same contradiction (via Remark 3).

Lemma 2. *Every element $g_j$ of $G$ has a unique representation $g_j = g_u x_u + g_v x_v$ for $x_u, x_v \in T$.*

*Proof.* By Lemma 1 we know that each $g_j$ has at least one such representation. Suppose $g_j = g_u\bar{x}_u + g_v\bar{x}_v = g_u x_u' + g_v x_v'$ where $\bar{x}_u, \bar{x}_v \in T$ and $x_u', x_v' \in T$. Write $r = \bar{x}_v - x_v'$ and $s = x_u' - \bar{x}_u$. Then $rg_v = sg_u$ and we may assume without loss that $r \geqslant 0$. Hence,

$$q > r \geqslant 0 \qquad \text{and} \qquad p^* > |s|.$$

*Case* 1. $s \geqslant 0$.

The relation $rg_v = sg_u$ together with $p^* > s$ and $q > r$ immediately contradicts the definition of $p^*$.

*Case* 2. $s \leqslant 0$ (and $s > -p^*$).

From $\bar{x}_u$, $\bar{x}_v \in T$ and $x_u'$, $x_v' \in T$ it follows that either $r < q_0$ or $s > -p_0$.

*Case* 2*a*. $r < q_0$. Let $s' = p^* + s$ and $r' = q^* + r$. Then $s'g_u = r'g_v$. Moreover, $0 < s' \leqslant p^*$ and (from $r < q_p = q - q^*$) $q^* \leqslant r' < q$. By Remarks 1 and 2 $r' = q^*$ and $s' = p^*$, and hence, $r = s = 0$.

*Case* 2*b*. $x > -p_0$. Let $s' = p - s$ and $r' = q - r$. As before $s'g_u = r'g_v$. Moreover, $p \leqslant s' < p^*$ and $0 < r' \leqslant q$. By Remarks 1 and 2 $s' = p$ and $r' = q$, and hence $r = s = 0$, completing the proof.

Several interesting observations follow from Lemmas 1 and 2.

*Remark* 4. $S_j = q\bar{x}_u + p\bar{x}_v$, where $\bar{x}_u$, $\bar{x}_v$ is the unique pair $x_u$, $x_v \in S_j \cap T$. (Note this implies $\delta_u = q$ and $\delta_v = p$.)

*Remark* 5. If $g_j = g_s - g_r = g_u\bar{x}_u + g_v\bar{x}_v$, and $\delta_s - \delta_r = q\bar{x}_u + p\bar{x}_v$ for $\bar{x}_u$, $\bar{x}_v \in T$, then $\delta_r + \delta_j = \delta_s$.

Remark 5 will be one of the key observations used to prove the coefficients $a_j$ actually determine a face. Now we verify the claim made earlier that the $\delta_j$ and $a_j$ are the same.

LEMMA 3. $\delta_j = a_j$ *for* $a_j$ *given by the rules of Section* 3.2.

*Proof.* First note that the rules of Section 3.2 assign each $a_j$ a value $a_j = qx_u + px_v$ where the pair $x_u$, $x_v$ satisfies $x_u$, $x_v$, $\in S_j$. We will further show that $x_u$, $x_v \in S_j \cap T$ thereby proving Lemma 3 by reference to Remark 4. This is clearly true for $a_j$ defined by $a_j = px_v$ (corresponding to $g_j = g_vx_v$) for $x_v = 0, 1,..., q - 1$. Moreover, values $a_j$ for all remaining $g_j$ are determined by the rule: For each $\bar{x}_v = 0, 1,..., q - 1$, let $a_j = q\bar{x}_u + p\bar{x}_v$ (corresponding to $g_j = g_u\bar{x}_u + g_v\bar{x}_v$) where $x_u = 0,..., k - 1$ and $k$ is the least positive integer such that $g_uk + g_v\bar{x}_v = g_vx_v$ for some $x_v$ satisfying $0 \leqslant x_v < q$. This rule must evidently assign values $a_j = q\bar{x}_u + p\bar{x}_v$ for all pairs $\bar{x}_u$, $\bar{x}_v$ in $T$. But, in fact, each $g_j$ is assigned a value $a_j$ exactly once by the rule, which completes the proof of Lemma 3 using Lemma 1.

Remark 6, to follow, shows the interesting fact that the smallest positive cost assigned to the 0 group element ($=p_0g_u + q_0g_v$) is actually equal to the order $(h + 1)$ of the group.

*Remark* 6.

$$p_0q + q_0p = h + 1.$$

*Proof.* We show that $p_0q + q_0p$ is equal to the number of pairs $x_u$, $x_v$ in $T$. There are $p^*$ values of $x_u$ from 0 to $p^* - 1$ that can combine with each of the $q_0$ values of $x_u$ from 0 to $q - 1$, giving $p^*q_0$ unique pairs $x_ux_v$. In addition, the $p_0$ values of $x_u$ from 0 to $p_0 - 1$ can combine with the $q^*$ values of $x_v$ from $q_0$ to $q - 1$, thus giving a total of $p^*q_0 + q^*p_0$ pairs in $T$. The remark follows by noting $p^* = p + p_0$ and $q^* = q - q_0$.

To complete the proof that the coefficients $a_j$ determine a face requires identifying the group element $\alpha_0$. We shall *define* a group element $g_s$ below and then prove that it is the element $\alpha_0$ we seek.

DEFINITION. $g_s = (q - 1) g_v - g_u$ (alternatively, $g_s = (p - 1) g_u - g_v$).

*Remark* 7.

$$g_s = (p^* - 1) g_u + (q_0 - 1) g_v$$

and

$$a_s = (p^* - 1) q + (q_0 - 1) = h + 1 + pq - (p + q).$$

*Proof.* The expression for $g_s$ follows directly from the definitions of $g_s$, $p^*$ and $q_0$. The first value for $a_s$ follows by noting that $x_u = p^* - 1$ and $x_v = q_0 - 1$ imply $x_u, x_v \in T$. The second value for $a_s$ follows from Remark 6.

*Remark* 8.

$$g_s = (p^* - 1 - p) g_u + (q_0 - 1 + q) g_v$$

and

$$a_s = (p^* - 1 - p) q + (q_0 - 1 + q) p$$

*Proof.* The remark follows from Remark 7 and $pg_u = qg_v$.

LEMMA 4. *For every $g_r \in G$ and $g_j = g_s - g_r : a_r + a_j = a_s$.*[5]

*Proof.* Suppose $\bar{x}_u, \bar{x}_v \in T \cap S_r$. Then $\bar{x}_u < p^*$, $\bar{x}_v < q$, and either $\bar{x}_v < q_0$ or $\bar{x}_u < p_0$. If $\bar{x}_v < q_0$ it follows from Remarks 4, 7, and Lemma 3 that $g_s - g_r = g_u x_u' + g_v x_v'$ and $a_s - a_r = q x_u' + p x_v'$ for $x_u', x_v' \in T$. Thus, by Remark 5, $a_s = a_r + a_j$. If, on the other hand, $\bar{x}_v > q_0$ (and hence $\bar{x}_u < p_0$), the same argument again applies invoking Remark 8 in place of Remark 7.

THEOREM. *For $p$ and $q$ and the coefficients $a_j$ given by the rules of Section* 3, *the inequality* (6) *is a face for the system* (5) *with $\alpha_0 = g_s$.*

*Proof.* That the inequality (6) is satisfied by every solution to (5) follows from Lemma 1 and Lemma 3. Also, by Lemma 4, for every $g_r \neq g_u, g_v, g_0$

[5] For completeness in this lemma we may define the "$a_j$ coefficient" corresponding to the "0" group element ($g_0$) to be 0.

there is a solution to (5) satisfying (6) with equality, in which $w_r = 1$ and $w_j = 0$ for all $j \neq u, v$. There are $h - 2$ such solutions, all clearly linearly independent. Finally, Remark 7 and Remark 8 provide linearly independent solutions involving only $w_u$ and $w_v$ which, together with the solutions already specified, give a total of $h$ linearly independent solutions satisfying (6) with equality. This proves the theorem.

## 5. Faces Obtained by Letting $g_v$ Vary

Having selected a generator $g_u$ for $G$, $g_v$ may range over all other elements of $G$ to generate faces in the manner indicated. It is immediately evident that if a second generator $g_w$ is selected, and $g_v$ again is permitted to range over the other elements of $G$, then the faces generated from $g_w$ will be homomorphically equivalent to those generated from $g_u$. Thus it suffices to select a single generator $g_u$ to implement this face generating procedure. However, given $g_u$, some of these faces obtained by varying $g_v$ will still be equivalent to others under homomorphism unless certain additional rules are followed. We conjecture that every face generated by these additional rules is unique for groups of prime order (in which every element is a generator), although some duplications apparently remain for other cyclic groups. The justification of the following rules is obvious, and hence is omitted.

Rule 1. All "first faces" are identical. Hence generate only one of them.

Rule 2. If $g_r$ is a generator for $G$, and $kg_r = g_u$, then the faces for $g_u$ and $g_v = g_r$ are equivalent to those for $g_u$ and $g_v = kg_u$. Hence bypass the latter after generating the former.

Rule 3. If $g_v = -g_u$, the faces for $q \leqslant h/2$ are the same as those for $q \geqslant h/2$; hence stop generating faces for $g_u$ and $g_v = -g_u$ when $q \leqslant h/2$.[6]

Example. The following table records the faces generated by the preceding rules for the group equation

$$1w_1 + 2w_2 + \cdots + 10w_{10} \equiv \alpha_0 \pmod{11}$$

using $g_1 = 1$ as the generator $g_u$.

[6] The faces generated when $g_v = -g_u$ appear to be equivalent to those generated by Gomory's "$\theta$-Method" [8].

TABLE 2

Faces Generated for the Cyclic Group of Order 11 Using the Generator $g_u = 1$

| $g_v$ | $g_t$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_0$ | $\lambda_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 10 |
| 5 | 4 | 3 | 6 | 9 | 12 | 4 | 7 | 10 | 13 | 16 | 8 | 16 | 9 |
| | 3 | 5 | 10 | 15 | 9 | 3 | 8 | 13 | 18 | 12 | 6 | 18 | 8 |
| | 2 | 7 | 14 | 10 | 6 | 2 | 9 | 16 | 12 | 3 | 4 | 16 | 7 |
| 7 | 3 | 2 | 4 | 6 | 8 | 10 | 12 | 3 | 5 | 7 | 9 | 12 | 6 |
| | 2 | 5 | 10 | 4 | 9 | 14 | 8 | 2 | 7 | 12 | 6 | 14 | 5 |
| 10 | 9 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 9 | 18 | 9 |
| | 8 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 16 | 8 | 24 | 8 |
| | 7 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 21 | 14 | 7 | 28 | 7 |
| | 6 | 5 | 10 | 15 | 20 | 25 | 30 | 24 | 18 | 12 | 6 | 30 | 6 |

TABLE 3

Faces Homomorphically Equivalent to those of Table 2 for $\alpha_0 = 10$

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_0$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 10 |
| 6 | 12 | 7 | 13 | 8 | 3 | 9 | 4 | 10 | 16 | 16 |
| 15 | 8 | 12 | 5 | 9 | 13 | 6 | 10 | 3 | 18 | 18 |
| 6 | 12 | 7 | 2 | 8 | 14 | 9 | 4 | 10 | 16 | 16 |
| 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 12 | 12 |
| 8 | 5 | 2 | 10 | 7 | 4 | 12 | 9 | 6 | 14 | 14 |
| 4 | 8 | 12 | 16 | 9 | 2 | 6 | 10 | 14 | 18 | 18 |
| 9 | 18 | 16 | 3 | 12 | 21 | 8 | 6 | 15 | 24 | 24 |
| 16 | 21 | 4 | 20 | 14 | 8 | 24 | 7 | 12 | 28 | 28 |
| 25 | 6 | 20 | 12 | 15 | 18 | 10 | 24 | 5 | 30 | 30 |

## 6. Computational Experience

A computer program has been written in Fortran IV to generate faces for the group equation (6) in the form[7]

$$1w_1 + 2w_2 + \cdots + hw_h \equiv \alpha_0 \pmod{h + 1}$$

where $\alpha_0$ is an integer, and the generator $g_u$ is selected to be 1.

Results are summarized in Tables 4 and 5 below.

[7] This form is general since all cyclic groups of the same order are isomorphic.

TABLE 4

Cyclic Groups of Prime Order

| (1) Group order | (2) Faces generated | (3) Total time* | (4) Time* per face | (5) Time* per coefficient |
|---|---|---|---|---|
| 31 | 65 | .036 | .00055 | .0000179 |
| 61 | 189 | .185 | .00098 | .0000160 |
| 151 | 707 | 1.655 | .00234 | .0000155 |
| 181 | 912 | 2.499 | .00274 | .0000151 |
| 211 | 1,129 | 3.647 | .00323 | .0000153 |
| 271 | 1,585 | 6.465 | .00408 | .0000150 |
| 331 | 2,079 | 10.371 | .00499 | .0000151 |
| 421 | 2,866 | 18.181 | .00634 | .0000151 |
| 541 | 3,989 | 32.172 | .00807 | .0000149 |
| 571 | 4,278 | 36.056 | .00843 | .0000148 |
| 601 | 4,578 | 40.884 | .00893 | .0000149 |
| 661 | 5,181 | 50.519 | .00975 | .0000148 |
| 691 | 5,488 | 55.816 | .01017 | .0000147 |
| 751 | 6,113 | 67.898 | .01111 | .0000148 |
| 811 | 6,751 | 80.856 | .01198 | .0000148 |
| 1,051 | 9,422 | 141.442 | .01501 | .0000143 |
| 1,471 | 14,453 | 302.923 | .02096 | .0000142 |

* Seconds of central processing time on the CDC 6600.

TABLE 5

Cyclic Groups Not of Prime Order

| (1) Group order | (2) Faces generated | (3) Total time* | (4) Time* per face | (5) Time* per coefficient |
|---|---|---|---|---|
| 30 | 78 | .032 | .00041 | .0000141 |
| 60 | 231 | .173 | .00075 | .0000127 |
| 150 | 910 | 1.686 | .00185 | .0000124 |
| 180 | 1,118 | 2.403 | .00215 | .0000120 |
| 210 | 1,441 | 3.633 | .00252 | .0000121 |
| 240 | 1,704 | 4.880 | .00286 | .0000120 |
| 270 | 2,029 | 6.600 | .00325 | .0000121 |
| 330 | 2,711 | 10.697 | .00395 | .0000120 |
| 420 | 3,630 | 17.794 | .00490 | .0000117 |
| 540 | 5,022 | 31.851 | .00634 | .0000118 |
| 570 | 5,718 | 38.745 | .00678 | .0000119 |
| 600 | 5,888 | 41.607 | .00707 | .0000118 |
| 630 | 6,241 | 45.963 | .00736 | .0000117 |
| 660 | 6,670 | 51.560 | .00773 | .0000117 |

* Seconds of central processing time on the CDC 6600.

From these tables it can be seen that the number of faces generated becomes a progressively larger multiple of the group order as the order increases.[8] Thus, for example, in Table 4 the multiple is roughly 2 when the order is 31, and is nearly 10 when the order is 1,471.

Because the method does not in general generate all faces, the number of faces recorded in column 2 of these tables is not the same as the total number of faces. Also, the number of faces recorded in Table 5 may be larger than the number of unique faces generated (i.e., those not equivalent to some other face under homomorphism). It is conjectured, however, that faces generated for prime order groups (Table 4) contain no duplications.

## 7. Conclusion

The face generating procedure succeeds in generating large numbers of faces very rapidly. The time to generate each coefficient of a face (total time divided by the product of one less than the group order and the number of faces generated) is only 15 microseconds for prime order groups and 12 microseconds for the other cyclic groups (slightly more for smaller groups and slightly less for larger ones). For some applications the method may generate "too many" faces. For example, the 14,453 faces for the group of order 1,471 is far more than the number of cuts anyone would reasonably wish to adjoin in a block to an integer program. But the groups associated with many integer programming problems have more than 1,471 elements, and for these the method will generate proportionately[9] more faces.

It would seem useful, therefore, to specify criteria of value for faces (or, more generally, sets of faces) and generate a select number of faces that satisfy these criteria. Such considerations are examined in [3].

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[8] The precise relationship between the group order and the number of faces generated by the method has not been determined.

[9] Actually, as seen, the proportion itself increases as the group order increases.

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